International Journal of Theoretical Physics, Vol. 37, No. 1, 1998

Equilibrium Statistical Ensembles and Structure of the Entropy Functional in Generalized Quantum Dynamics

Stephen L. Adler[1] and L. P. Horwitz[1,2]

Received July 4, 1997

We review the microcanonical and canonical ensembles constructed on an underlying generalized quantum dynamics and the algebraic properties of the conserved quantities. We discuss the structure imposed on the microcanonical entropy by the equilibrium conditions.

1. INTRODUCTION

In this paper we review briefly the generalized quantum dynamics (Adler, 1994, 1995) constructed on a phase space of local noncommuting fields. We show that the equilibrium conditions on the microcanonical entropy imply that the system decomposes thermodynamically to a sequence of adiabatically independent subsystems, each with its own temperature. There is an equipartition theorem for the phase space variables of the system generated by the linear combination of conserved quantities associated with each of these independent thermodynamic modes.

We start with a review of our basic framework. Generalized quantum dynamics (Adler, 1994, 1995) is an analytic mechanics on a symplectic set of operator-valued variables, forming an operator-valued phase space S. These variables are defined as the set of linear transformations[3] on an underlying real, complex, or quaternionic Hilbert space (Hilbert module), for which the postulates of a real, complex, or quaternionic quantum mechanics are satisfied

[1] School of Natural Sciences, Institute for Advanced Study, Princeton, New Jersey 08540.
[2] On Sabbatical leave from School of Physics, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Ramat Aviv 67798, Israel, and Department of Physics, Bar Ilan University, Ramat Gan 52900, Israel.
[3] In general, local (noncommuting) quantum fields.

0020-7748/98/0100-0519$15.00/0

(Adler, 1995; Stueckelberg, 1960, 1961, 1962; Finkelstein *et al.*, 1962, 1963; Horwitz and Biedenharn, 1984; Piron, 1976). The dynamical (generalized Heisenberg) evolution, or flow, of this phase space is generated by the total trace Hamiltonian $\mathbf{H} = \mathbf{Tr}H$, where for any operator O we have

$$\begin{aligned} \mathbf{O} \equiv \mathbf{Tr} &\equiv \mathrm{ReTr}(-1)^F \mathrm{O} \\ &= \mathrm{Re} \sum_n \langle n|(-1)^F \mathrm{O}|n\rangle \end{aligned} \tag{1.1}$$

H is a function of the operators $\{q_r(t)\}$, $\{p_r(t)\}$, $r = 1, 2, \ldots, N$ (realized as a sum of monomials, or a limit of a sequence of such sums; in the general case of local noncommuting fields, the index r contains continuous variables), and $(-1)^F$ is a grading operator with eigenvalue 1 (−1) for states in the boson (fermion) sector of the Hilbert space. Operators are called bosonic or fermionic in type if they commute or anticommute, respectively, with $(-1)^F$; for each r, p_r and q_r are of the same type.

The variation of a total trace functional with respect to some operator is defined with the help of the cyclic property of the **Tr** operation. The variation of any monomial O consists of terms of the form $\mathrm{O}_L \delta x_r \mathrm{O}_R$, for x_r one of the $\{q_r\}$, $\{p_r\}$, which, under the **Tr** operation, can be brought to the form

$$\delta\mathbf{O} = \delta\mathbf{Tr}\mathrm{O} = \pm\mathbf{Tr}\mathrm{O}_R\mathrm{O}_L\delta x_r$$

so that sums and limits of sums of such monomials permit the construction of

$$\delta\mathbf{O} = \mathbf{Tr} \sum_r \frac{\delta\mathbf{O}}{\delta x_r} \delta x_r \tag{1.2}$$

uniquely defining $\delta\mathbf{O}/\delta x_r$.

Assuming the existence of a total trace Lagrangian (Adler, 1994, 1995) $\mathbf{L} = \mathbf{L}(\{q_r\}, \{\dot{q}_r\})$, the variation of the total trace action

$$\mathbf{S} = \int_{-\infty}^{\infty} \mathbf{L}(\{q_r\}, \{\dot{q}_r\})\, dt \tag{1.3}$$

results in the operator Euler–Lagrange equations

$$\frac{\delta\mathbf{L}}{\delta q_r} - \frac{d}{dt}\frac{\delta\mathbf{L}}{\delta\dot{q}_r} = 0 \tag{1.4}$$

As in classical mechanics, the total trace Hamiltonian is defined as a Legendre transform,

$$\mathbf{H} = \mathbf{Tr} \sum_r p_r\dot{q}_r - \mathbf{L} \tag{1.5}$$

where

$$p_r = \frac{\delta \mathbf{L}}{\delta \dot{q}_r} \tag{1.6}$$

It then follows from (1.4) that

$$\frac{\delta \mathbf{H}}{\delta q_r} = -\dot{p}_r, \qquad \frac{\delta \mathbf{H}}{\delta p_r} = \varepsilon_r \dot{q}_r \tag{1.7}$$

where $\varepsilon_r = 1\ (-1)$ according to whether p_r, q_r are of bosonic (fermionic) type.

Defining the generalized Poisson bracket

$$\{\mathbf{A}, \mathbf{B}\} = \mathbf{Tr} \sum_r \varepsilon_r \left(\frac{\delta \mathbf{A}}{\delta q_r} \frac{\delta \mathbf{B}}{\delta p_r} - \frac{\delta \mathbf{B}}{\delta q_r} \frac{\delta \mathbf{A}}{\delta p_r} \right) \tag{1.8a}$$

one sees that

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + \{\mathbf{A}, \mathbf{H}\} \tag{1.8b}$$

Conversely, if we define

$$\mathbf{x}_s(\eta) = \mathbf{Tr}(\eta x_s) \tag{1.9a}$$

for η an arbitrary, constant operator (of the same type as $\mathbf{x}_s$, which denotes here q_s or p_s), then

$$\frac{d\mathbf{x}_s(\eta)}{dt} = \mathbf{Tr} \sum_r \varepsilon_r \left(\frac{\delta \mathbf{x}_s(\eta)}{\delta q_r} \frac{\delta \mathbf{H}}{\delta p_r} - \frac{\delta \mathbf{H}}{\delta q_r} \frac{\delta \mathbf{x}_s(\eta)}{\delta p_r} \right) \tag{1.9b}$$

and comparing the coefficients of η on both sides, one obtains the Hamilton equations (1.7) as a consequence of the Poisson bracket relation (1.8b).

The Jacobi identity is satisfied by the Poisson bracket (1.8a) (Adler *et al.*, 1994), and hence the total trace functionals have many of the properties of the corresponding quantities in classical mechanics (Adler and Wu, 1994). In particular, canonical transformations take the form

$$\delta \mathbf{x}_s(\eta) = \{\mathbf{x}_s(\eta), \mathbf{G}\} \tag{1.10a}$$

which implies that

$$\delta p_r = -\frac{\delta \mathbf{G}}{\delta q_r}, \qquad \delta q_r = \varepsilon_r \frac{\delta \mathbf{G}}{\delta p_r} \tag{1.10b}$$

with the generator $\mathbf{G}$ any total trace functional constructed from the operator phase space variables. Time evolution then corresponds to the special case $\mathbf{G} = \mathbf{H}\,dt$.

It has recently been shown by Adler and Millard (1996) that a canonical ensemble can be constructed on the phase space $\mathcal{S}$, reflecting the equilibrium properties of a system of many degrees of freedom. Since the operator

$$\begin{aligned}\tilde{C} &= \sum_r (\varepsilon_r q_r p_r - p_r q_r) \\ &= \sum_{r,B} [q_r, p_r] - \sum_{r,F} \{q_r, p_r\} \qquad (1.11)\end{aligned}$$

where the sums are over bosonic and fermionic pairs, respectively, is conserved under the evolution (1.7) induced by the total trace Hamiltonian, the canonical ensemble must be constructed taking this constraint into account. This is done by including in the canonical exponent the conserved quantity $\mathbf{Tr}\tilde{\lambda}\tilde{\mathbb{C}}$, for some given constant anti-Hermitian operator $\tilde{\lambda}$.

In the general case, in the presence of the fermionic sector, the graded trace of the Hamiltonian is not bounded from below, and the partition function may be divergent. When the equations of motion induced by the Lagrangian $\mathbf{L}$ coincide with those induced by the ungraded total trace of the same Lagrangian,

$$\hat{\mathbf{L}} = \mathrm{ReTr}L$$

without the factor $(-1)^F$, the corresponding ungraded total trace Hamiltonian $\hat{\mathbf{H}}$ is conserved; it may therefore be included as a constraint functional in the canonical ensemble, along with the new conserved quantity $\hat{\mathbf{Tr}}\hat{\tilde{\lambda}}\hat{\tilde{\mathbb{C}}}$ (see Appendices 0 and C of Adler and Millard, 1996) where

$$\begin{aligned}\hat{\tilde{C}} &= \sum_r [q_r, p_r] \\ &= \sum_{r,B} [q_r, p_r] + \sum_{r,F} \{q_r, p_r\} \qquad (1.12)\end{aligned}$$

It was argued that the Ward identities derived from the canonical ensemble imply that λ and $\tilde{\lambda}$ are functionally related, so that they may be diagonalized in the same basis (Appendix F of Adler and Millard, 1996). It was then shown that, since the ensemble averages depend only on $\tilde{\lambda}$ and $(-1)^F$, the ensemble average of any operator must commute with these operators. Since the ensemble-averaged operator $\langle\tilde{C}\rangle_{AV}$ is anti-self-adjoint, if one furthermore assumes it is completely degenerate (with eigenvalue $i_{\mathrm{eff}}\hbar$), the ensemble average of the theory then reduces to the usual complex quantum field theory.

As discussed in detail in Adler and Horwitz (1996), the phase-space volume associated with the microcanonical ensemble can be written as

$$\begin{aligned}\Gamma(E, \hat{E}, \tilde{\nu}, \hat{\tilde{\nu}}) = \int d\mu\, \delta(E - \mathbf{H})\delta(\hat{E} - \hat{\mathbf{H}}) \\ \times \prod_{n\le m,\, A} \delta(\tilde{\nu}^A_{nm} - \langle n|(-1)^F\tilde{C}|m\rangle^A)\delta(\hat{\tilde{\nu}}^A_{nm} - \langle n|\hat{\tilde{C}}|m\rangle^A) \qquad (1.13)\end{aligned}$$

where we have taken into account the possible algebraic structure of the matrix elements of the operators with the index A, which takes the values 0, 1 for the complex Hilbert space, 0, 1, 2, 3 for the quaternionic Hilbert space, and just the value 0 for real Hilbert space. The invariant phase space measure is defined by

$$d\mu = \prod_A d\mu^A \tag{1.14}$$

$$d\mu^A \equiv \prod_{r,m,n} d(x_r)^A_{mn}$$

where redundant factors are omitted according to adjointness conditions. We have, furthermore, used the abbreviations $\tilde{\nu} \equiv \{\nu^A_{nm}\}$ and $\hat{\tilde{\nu}} \equiv \{\hat{\nu}^A_{nm}\}$. The entropy associated with this ensemble is given by

$$S_{\mathrm{mic}}(E, \hat{E}, \tilde{\nu}, \hat{\tilde{\nu}}) = \ln \Gamma(E, \hat{E}, \tilde{\nu}, \hat{\tilde{\nu}}) \tag{1.15}$$

It was argued in Adler and Horwitz (1996) that a large system can be decomposed into a part within a certain (large) region of the measure space, which we denote as b, corresponding to what we shall consider as a *bath*, in the sense of statistical mechanics, and another (small) part which we shall denote as s, corresponding to what we shall consider as a *subsystem.* It was then argued that the phase-space volume can be well approximated by

$$\Gamma(E, \hat{E}, \tilde{\nu}, \hat{\tilde{\nu}}) = \int dE_s d\hat{E}_s (d\nu^s)(d\hat{\nu}^s) \tag{1.16}$$
$$\times \Gamma_b(E - E_s, \hat{E} - \hat{E}_s, \tilde{\nu} - \tilde{\nu}_s, \hat{\tilde{\nu}} - \hat{\tilde{\nu}}_s)\Gamma_s(E_s, \hat{E}_s, \tilde{\nu}_s, \hat{\tilde{\nu}}_s)$$

Defining the variables

$$\xi = \{\xi_i\} \equiv \{E, \hat{E}, \tilde{\nu}, \hat{\tilde{\nu}}\} \tag{1.17}$$

it was shown (Adler and Horwitz, 1996) that the equilibrium conditions which follow from the assumption that there is a maximum in the integrand of (1.16) (which dominates the integral in the limit of a large number of degrees of freedom) result in the set of equalities

$$\frac{1}{\Gamma_s(\xi)} \frac{\partial \Gamma_s}{\partial \xi_i}(\xi)|_{\bar{\xi}} = \frac{1}{\Gamma_b(\Xi - \xi)} \frac{\partial \Gamma_b}{\partial \Xi_i}(\Xi - \xi)|_{\bar{\xi}} \tag{1.18}$$

where Ξ corresponds to the total quantities belonging to the full system. It was then shown that the canonical ensemble obtained by Adler and Millard (1996)

$$\rho = Z^{-1} \exp - \{\tau \mathbf{H} + \hat{\tau}\hat{\mathbf{H}} + \mathbf{Tr}\tilde{\lambda}\tilde{C} + \hat{\mathbf{Tr}}\hat{\tilde{\lambda}}\hat{\tilde{C}}\} \tag{1.19}$$

where

$$Z = \int d\mu \exp - \{\tau \mathbf{H} + \hat{\tau}\hat{\mathbf{H}} + \mathbf{Tr}\tilde{\lambda}\tilde{C} + \hat{\mathbf{Tr}}\hat{\tilde{\lambda}}\hat{\tilde{C}}\} \tag{1.20}$$

follows in a straightforward way. The quantities τ, $\hat{\tau}$ and the matrices (real, complex, or quaternionic) λ, $\tilde{\lambda}$ are the equilibrium parameters defined by the values of the members of (1.18) for each of the ξ's (Adler and Horwitz, 1996); they therefore correspond to *temperatures* precisely as they emerge in conventional statistical mechanics. We remark that Ingarden (1968; see also Ingarde and Kossakowski, 1986, and Ingarden, 1979) has studied a similar generalization of temperature in the framework of the statistical mechanics associated with problems of optical pumping (in the diagonal form which we shall discuss in the next section).

Replacing the operators and trace functionals in (1.20) by integrals over δ-functions, the partition function can be rewritten as (Adler and Horwitz, 1996)

$$Z = \int dE\, d\hat{E}(d\nu)(d\hat{\nu}) \exp\{S_{\text{mic}}(E, \hat{E}, \tilde{\nu}, \hat{\tilde{\nu}})\}$$

$$\times \exp - \{\tau E + \hat{\tau}\hat{E} + \mathbf{Tr}\tilde{\lambda}\tilde{\nu} + \hat{\mathbf{Tr}}\hat{\tilde{\lambda}}\hat{\tilde{\nu}}\} \tag{1.21}$$

By studying the dispersions of the variables in the canonical ensemble, it was found (Adler and Horwitz, 1996) that the second derivative matrix of the microcanonical entropy is negative definite, i.e., that

$$\left(\frac{\partial^2 S_{\text{mic}}}{\partial \xi_i \partial \xi_j} \right) \le 0 \tag{1.22}$$

In the following we use the fact that this matrix is real symmetric to diagonalize it, and in this way to construct a set of dynamical generators over which the total entropy decomposes in a neighborhood C_0 of the maximum entropy point.

2. DIAGONAL FORM OF THE SECOND VARIATION OF THE ENTROPY

The negative-definite matrix (1.22)

$$D_{ij} = \frac{\partial^2 S_{\text{mic}}}{\partial \xi_i \partial \xi_j} \tag{2.1}$$

is symmetric and can therefore be diagonalized by an orthogonal transformation. Let a_{ij} (orthogonal) be such that, in the neighborhood C_0,

$$\sum_{ij} a_{ki} a_{lj} D_{ij} = \delta_{kl} d_l(\xi) \tag{2.2}$$

where the elements $d_l(\xi)$ on the right-hand side are the negative eigenvalues. Now, let us define, using these constant coefficients,

$$e_k = \sum_i a_{ki} \xi_i \tag{2.3a}$$

and hence

$$\xi_i = \sum_k a_{ki} e_k \tag{2.3b}$$

It then follows that, in C_0,

$$\begin{aligned} \frac{\partial^2 S}{\partial e_k \partial e_l} &= \sum_{ij} a_{ki} a_{lj} \frac{\partial^2 S}{\partial \xi_i \partial \xi_j} \\ &= \sum_{ij} a_{ki} a_{lj} D_{ij} = \delta_{kl} d_l(\xi) \end{aligned} \tag{2.4}$$

Since the crossed derivatives of S vanish, S must be a sum of functions that depend on each of the $\{e_k\}$ separately, i.e.,

$$S = \sum_k S_k(e_k) \tag{2.5}$$

The entropy is therefore additive (in C_0) over diagonal "thermodynamic modes."

The equilibrium parameters defined in the previous section,

$$\chi_j = \frac{\partial S}{\partial \xi_j} = \{\tau, \hat{\tau}, \lambda, \hat{\lambda}\} \tag{2.6}$$

may be transformed in the same way, i.e.,

$$\begin{aligned} \sum_j a_{kj} \chi_j &= \sum_j a_{kj} \frac{\partial}{\partial \xi_j} S = \frac{\partial S}{\partial e_k} \\ &= \frac{\partial S_k(e_k)}{\partial e_k} \equiv \frac{1}{T_k} \end{aligned} \tag{2.7}$$

giving the diagonal temperatures [of the type considered by Ingarden (1968)].

We remark that, according to (2.2) and (2.4),

$$\frac{\partial^2 S}{\partial e_k^2} = d_k < 0 \tag{2.8}$$

so that the "specific heats," entering as

$$\frac{\partial}{\partial e_k}\frac{1}{T_k} = -\frac{1}{T_k^2}\frac{dT_k}{de_k} = -\frac{1}{T_k^2}\frac{1}{C_k} \tag{2.9}$$

are positive, and by (2.7)–(2.9) are given by

$$C_k = -\frac{1}{T_k^2} d_k \tag{2.10}$$

3. EQUIPARTITION

We now consider linear combinations of the dynamical quantities

$$\mathcal{H}_i = \{\mathbf{H}, \hat{\mathbf{H}}, \tilde{C}, \hat{\tilde{C}}\}$$

of the same form as the linear combinations of the parameters $\{\xi_i\}$ which are their equilibrium values,

$$\varepsilon_k = \sum_i a_{ki}\mathcal{H}_i \tag{3.1}$$

the effective "energies" associated with the thermodynamic modes. Since the determinant of the matrix a is unity, the microcanonical phase space integral (1.13) can be written as

$$\Gamma(\xi) = \int d\mu \prod_k \delta(e_k - \varepsilon_k) \tag{3.2}$$

Since, however, as we have shown in Section 2,

$$\begin{aligned} \ln \Gamma(\xi) &= S(e_1, e_2, \ldots) \\ &= \sum_k S_k(e_k) \end{aligned} \tag{3.3}$$

it follows that the phase-space volume factorizes on the diagonal parameters

$$\begin{aligned} \Gamma(\xi) &= \exp\left\{\sum_k S_k(e_k)\right\} = \prod_k \exp\{S_k(e_k)\} \\ &= \Gamma(e_1, e_2, \ldots) \equiv \prod_k \Gamma_k(e_k) \end{aligned} \tag{3.4}$$

One can show that the free energy also becomes additive (Adler and Horwitz, n.d.).

Let us now consider the microcanonical average

$$\left\langle q_r \frac{\delta\varepsilon_k}{\delta q_s}\right\rangle = \frac{1}{\Gamma(e_1, e_2, \ldots)} \int d\mu \prod_l \delta(e_l - \varepsilon_l)\, q_r \frac{\delta\varepsilon_k}{\delta q_s} \tag{3.5}$$

where $\{q_r\}$ are the canonical coordinates (fields) of the phase space. We now write the right-hand side of (3.5) identically as

$$\frac{1}{\Gamma(e_1, e_2, \ldots)} \prod_l \frac{\partial}{\partial e_l} \int_{\{\varepsilon_j < e_j\}} d\mu \; q_r \frac{\delta}{\delta q_s} (\varepsilon_k - e_k)$$

replacing the δ-functions by derivatives of the parameters of boundary step functions; adding the constant e_k does not affect the result. Integrating by parts in the integration over phase space, we obtain

$$\left\langle q_r \frac{\delta \varepsilon_k}{\delta q_s} \right\rangle = \frac{1}{\Gamma(e_1, e_2, \ldots)} \prod_l \frac{\partial}{\partial e_l} \int_{\varepsilon_j < e_j} \left\{ \frac{\delta}{\delta q_s} q_r(\varepsilon_k - e_k) - \delta_{rs}(\varepsilon_k - e_k) \right\} \tag{3.6}$$

The first term vanishes on the boundary, and we therefore have

$$\left\langle q_r \frac{\delta \varepsilon_k}{\delta q_s} \right\rangle = -\frac{\delta_{rs}}{\Gamma(e_1, e_2, \ldots)} \prod_l \frac{\partial}{\partial e_l} \int_{\{\varepsilon_j < e_j\}} d\mu(\varepsilon_k - e_k) \tag{3.7}$$

The derivative with respect to e_k in the product of derivatives vanishes when it differentiates the upper bound; its contribution is only from the integrand, resulting in a factor −1. The other derivatives act only on the upper limits. The product then results in the restricted measure

$$\int_{\varepsilon_k < e_k} d\mu(\varepsilon_l = e_l \; \forall l \neq k) \equiv \Sigma_k \tag{3.8}$$

which can be rewritten as

$$\Sigma_k = \int_{\{\varepsilon_j < e_j\}} d\mu \prod_{l \neq k} \delta(\varepsilon_l - e_l) \tag{3.9}$$

According to (3.2),

$$\frac{\partial \Sigma_k}{\partial e_k} = \Gamma(e_1, e_2, \ldots) \tag{3.10}$$

We therefore have

$$\left\langle q_r \frac{\delta \varepsilon_k}{\delta q_s} \right\rangle = \frac{\delta_{rs}}{\Gamma(e_1, e_2, \ldots)} \Sigma_k \tag{3.11}$$

We now use the factorization of $\Gamma(e_1, e_2, \ldots)$ in C_0 to derive a relation between Σ_k and the additive entropies. In the limit of a large number of degrees of freedom, the leading edge of the integral defining Σ_k dominates the integral (Huang, 1987), so we may formally extrapolate, as a model, the quadratic form (and associated factorization) valid in C_0. From (3.4) and (3.10) it then follows that

$$\frac{\partial \Sigma_k}{\partial e_k} = \Gamma_k(e_k) \prod_{l \neq k} \Gamma_l(e_l) \tag{3.12}$$

One may integrate this equation to obtain

$$\Sigma_k = \int^{e_k} \Gamma_k(e'_k) de'_k \prod_{l \neq k} \Gamma_l(e_l) + G(e_l, l \neq k) \tag{3.13}$$

Since ε_k cannot be $-\infty$ (the functional $\hat{\mathbf{H}}$ is contained linearly and its positive values are assumed to dominate for large values of the phase space variables), the first term on the right-hand side of (3.13) along with Σ_k must vanish as $e_k \to -\infty$, and hence G must be zero.

We therefore obtain

$$\Sigma_k = \int^{e_k} de'_k e^{S_k(e'_k)} \prod_{l \neq k} e^{S_l(e_l)} \tag{3.14}$$

so that

$$\begin{aligned} \frac{\Sigma_k}{\Gamma(e_1, e_2, \ldots)} &= \frac{\int^{e_k} de'_k e^{S_k(e'_k)}}{e^{S_k(e_k)}} \\ &= \frac{1}{(d/de_k) \ln \int^{e_k} de'_k \, e^{S_k(e'_k)}} \end{aligned} \tag{3.15}$$

With the leading approximation for a large number of degrees of freedom (Huang, 1987)

$$\ln \int^{e_k} de'_k \, e^{S_k(e'_k)} \sim \ln e^{S_k(e_k)}$$

we conclude that

$$\left\langle q_r \frac{\delta \varepsilon_k}{\delta q_s} \right\rangle = -\delta_{rs} T_k \tag{3.16}$$

We finally make some remarks on the flows generated by ε_k, which, for clarity, we recast to the form (summed on nm)

$$\begin{aligned} \varepsilon_k &= a_{k0}\mathbf{H} + a_{k1}\hat{\mathbf{H}} + a_{k(mn)}C_{nm} + \hat{a}_{k(mn)}\hat{C}_{nm} \\ &= a_{k0}\mathbf{H} + a_{k1}\hat{\mathbf{H}} + \mathbf{Tr}(\tilde{a}_k\tilde{C}) + \mathbf{Tr}(\tilde{\hat{a}}_k\tilde{\hat{C}}) \end{aligned} \tag{3.17}$$

The Poisson bracket (1.8a) then contains a term which is the t-derivative, by (1.9b), but there are additional terms of general type (1.10b). In Adler and Horwitz (1996) it is shown that the terms in (3.17) which contain $\tilde{C}$, $\hat{C}$ induce transformations on phase space which are commutators with $\tilde{a}_k$, $\hat{a}_k$ in the boson sector, and with $\tilde{a}_k$ in the fermion sector, but anticommutators with $\hat{a}_k$ in the fermionic sector. Hence the elements of the diagonalization transformation act as connection forms under evolution generated by the effective mode energy functionals. Further discussion and application of these results will be given in Adler and Horwitz (n.d.).

ACKNOWLEDGMENTS

This work was supported by the Department of Energy under Grant DE-FG02-90ER40542. One of us (L.P.H.) wishes to thank C. Piron and D. Moore for discussions, and for directing us to the work of Ingarden (1968).

REFERENCES

Adler, S. L. (1994). *Nuclear Physics B,* **415**, 195.
Adler, S. L. (1995). *Quaternionic Quantum Mechanics and Quantum Fields*, Oxford University Press, Oxford.
Adler, S. L., and Horwitz, L. P. (1996). *Journal of Mathematical Physics*, **37**, 5429.
Adler, S. L., and Horwitz, L. P. (n.d.). In preparation.
Adler, S. L., and Millard, A. C. (1996). *Nuclear Physics B*, **473**, 199.
Adler, S. L., and Wu, Y.-S. (1994). *Physical Review D,* **49**, 6705.
Adler, S. L., Bhanot, G. V., and Weckel, J. D. (1994). *Journal of Mathematical Physics,* **35**, 531.
Finkelstein, D., Jauch, J. M., Shiminovich, S., and Speiser, D. (1962). *Journal of Mathematical Physics,* **3**, 207.
Finkelstein, D., Jauch, J. M., Shiminovich, S., and Speiser, D. (1963). *Journal of Mathematical Physics,* **4**, 788.
Horwitz, L. P., and Biedenharn, L. C. (1984). *Annals of Physics,* **157**, 432.
Huang, K. (1987). *Statistical Mechanics*, Wiley, New York, 1987.
Ingarden, R. S. (1968). *Annales de l'Institute Henri Poincaré,* **8**(1), 1.
Ingarden, R. S. (1979). *Termodynamika Statystyczna*, Uniwersytet Mikolaja Kopernica, Skrypty i Teksty Pomocnicze, Toruń.
Ingarden, R. S., and Kossakowski, A. (1986). *Reports on Mathematical Physics,* **24**, 177.
Piron, C. (1976). *Foundations of Quantum Physics*, Benjamin, Reading, Massachusetts.
Stueckelberg, E. C. G. (1960). *Helvetica Physica Acta,* **33**, 727.
Stueckelberg, E. C. G. (1961). *Helvetica Physica Acta,* **34**, 621.
Stueckelberg, E. C. G. (1962). *Helvetica Physica Acta,* **35**, 673.